K25P 1138

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.S.S. – Supple./Imp.) Examination, April 2025
(2021 and 2022 Admissions)
MATHEMATICS
MAT 4E 02 – Fourier and Wavelet Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this part. **Each** question carries **4** marks.

1. Suppose $z, w \in l^2(\mathbb{Z}_N)$. For $k \in \mathbb{Z}$, prove that $z * \tilde{w}(k) = \langle z, R_k w \rangle$.
2. Define p^{th} stage wavelet filter sequence.
3. Define upsampling operator on $\mathbb{Z}$.
4. Define system matrix of two vectors in $l^2(\mathbb{Z})$.
5. Define the Fourier transform of the function $f \in L'(\mathbb{R})$.
6. Give an inner product on $L^2(\mathbb{R})$. **(4×4=16)**

PART – B

Answer **any four** questions from this part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Suppose $M \in \mathbb{N}$, $N = 2M$ and $w \in l^2(Z_N)$. Then prove that $\{R_{2k}w\}_{k=0}^{M-1}$ is an orthonormal set with M elements if and only if

$$\left|\hat{w}(n)\right|^2 + \left|\hat{w}(n+M)\right|^2 = 2 \text{ for } n = 0, 1 \ldots, M-1 \cdot$$

b) State and prove the folding lemma for vectors in $l^2(Z_N)$.

P.T.O.

8. a) Suppose $N = 2^n$, $1 \le p \le n$ and u_1, v_1, u_2, v_2, ... , u_p, v_p form a p^{th} stage wavelet filter sequence. Suppose $z \in l^2(Z_N)$. Then prove that the output $\{x_1, x_2, x_3,...,x_p, y_p\}$ of the analysis phase of the corresponding p^{th} stage wavelet filter bank can be computed using no more than $4N + N \log_2 N$ complex multiplications.

   b) Suppose N is divisible by 2^p. Suppose $u,v \in l^2(\mathbb{Z}_N)$ are such that the system matrix A(n) is unitary for all n. Let u_1=u and v_1= v and for all l = 2, 3,..., p, define u_1 and v_1 by the equations.

$$u_l(n) = \sum_{k=0}^{2^{l-1}-1} u_1\left(n + \frac{kN}{2^{l-1}}\right)$$

and

$$v_l(n) = \sum_{k=0}^{2^{l-1}-1} v_1\left(n + \frac{kN}{2^{l-1}}\right)$$

Then prove that u_1, v_1, u_2, v_2,...., u_p,v_p is a p-th stage wavelet filter sequence.

9. a) Suppose N is divisible by 2^l, $g_{l-1} \in l^2(\mathbb{Z}_N)$ and the set

$$\left\{R_{2^{l-1}k}\, g_{l-1}\right\}_{k=0}^{(N/2^{l-1})-1}$$

is orthonormal and has $N/2^{l-1}$ elements. Suppose u_l, $v_l \in l^2(\mathbb{Z}_{N/2^{l-1}})$ and the system matrix $A_l(n)$ is unitary for all n = 0, 1, ..., $(N/2^l)-1$. Define

$f_l = g_{l-1} * U^{l-1}(v_1)$ and $g_l = g_{l-1} * U^{l-1}(u_1)$.

Define spaces

$$V_{-l+1} = \text{span}\left\{R_{2^{l-1}k}\, g_{l-1}\right\}_{k=0}^{(N/2^{l-1})-1}$$

$$W_{-l} = \text{span}\left\{R_{2^l k} f_l\right\}_{k=0}^{(N/2^l)-1}$$

and $V_{-l} = \text{span}\left\{R_{2^l k} g_l\right\}_{k=0}^{(N/2^l)-1}$

Then prove that $V_{-l} \oplus W_{-l} = V_{-l+1}$.

   b) Suppose N is divisible by 2^l, $x,y,w \in l^2(Z_{N/2l})$, and $z \in l^2(Z_N)$. Then prove that $D^l(z) * w = D^l(z * U^l(w))$ and $U^l(x*y) = U^l(x) * U^l(y)$.

Unit – II

10. Suppose $\{a_j\}_{j\in Z}$ is an orthonormal set in a Hilbert space H and let

$$S = \left\{ \sum_{j\in Z} z(j)a_j : z = (z(j))_{j\in Z} \in l^2(Z) \right\}.$$

Define $P_s(f) = \sum_{j\in Z} \langle f, a_j \rangle a_j$.

Then prove the following :

a) For every $f \in H$, $P_s(f) \in S$;

b) The transformation $P_s : H \to S$ is linear ;

c) If $s \in S$, then $P_s(s) = s$;

d) $\langle f - P_s(f), s\rangle = 0$ for any $f \in H$ and $s \in S$.

11. a) If f and g are members of $L^2([-\pi,\pi))$, then prove that $f.g \in L'([-\pi,\pi))$.

b) Suppose $\theta_0 \in (-\pi,\pi)$ and $\alpha > 0$ is sufficiently small that $-\pi < \theta_0 - \alpha < \theta_0 + \alpha < \pi$.

Define intervals

$I = (\theta_0 - \alpha, \theta_0 + \alpha)$,

and $J = (\theta_0 - \alpha/2, \theta_0 + \alpha/2)$.

Then prove that there exists $\delta > 0$ and a sequence of real-valued trigonometric polynomials $\{p_n(\theta)\}_{n=1}^{\infty}$ such that

i) $p_n(\theta) \geq 1$ for $\theta \in I$

ii) $p_n(\theta) \geq (1+\delta)^n$ for $\theta \in J$

iii) $|p_n(\theta)| \leq 1$ for $\theta \in [-\pi,\pi)\backslash I$.

12. a) Suppose $f \in L'([-\pi,\pi))$ and

$$\left\langle f, e^{in\theta} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = 0 \text{ for all } n \in Z.$$

Then prove that f (θ) = 0 a.e.

b) Suppose H is a Hilbert space, $\{a_j\}_{j\in Z}$ is an orthonormal set in H and $f \in H$.

Then prove that the sequence $\{\langle f, a_j\rangle\}_{j\in Z}$ belongs to $l^2(\mathbb{Z})$ with

$$\sum_{j\in Z} \left|\left\langle f, a_j \right\rangle\right|^2 \leq \|f\|^2.$$

Unit – III

13. a) By means of examples, show that there is no containment between $L'(\mathbb{R})$ and $L^2(\mathbb{R})$.

b) Let $g \in L'(\mathbb{R})$ and $t > 0$. Then prove that $g_t : \mathbb{R} \to \mathbb{C}$ defined by

$$g_t(x) = \frac{1}{t} g\left(\frac{x}{t}\right)$$

is an element in $L'(\mathbb{R})$ and $||g_t||_1 = ||g||_1$.

14. a) If $f \in L'(\mathbb{R})$, then show that

$$\left|\int_{\mathbb{R}} f(x)dx\right| \leq \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1$$

b) Suppose $f, g \in L'(\mathbb{R})$. Then prove that $f * g \in L'(\mathbb{R})$ with

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

15. a) Prove that the trigonometric system is complete in $L^2([-\pi, \pi))$.

b) Suppose $f, g \in L^2(\mathbb{R})$ and $y \in \mathbb{R}$. Then prove that $\langle f, R_y g \rangle = (f * \tilde{g})(y)$. **(4×16=64)**

KANNUR UNIVERSITY